

# Generalized Orbits-Fixedpoints Relations

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## Abstract

I prove the following equality for  $t$ -transitive groups  $G$

$$\frac{1}{|G|} \sum_{g \in G} f_{\mathbb{Z}_N}(g)^k = \mathcal{N}_{orbits}(G, \mathbb{Z}_N^k) = \sum_{j=1}^{\min(k, N)} d_j(G) S(k, j),$$

where  $S(k, j)$  are Stirling numbers of the second kind, and  $d_j(G) = 1$  for  $1 \leq j \leq t$ , and  $d_j(G) \geq 2$  for  $j > t$ . The above equality extends further, using new proofs, two generalizations of an earlier fixedpoints-orbits theorem for finite group actions  $(S_N, \mathbb{Z}_N^k)$ . An illustration using Mathieu group  $M_{24}$  is discussed. Possible applications using tensor products of matrix permutation representations is indicated.

*Keywords:* group action, orbits, fixed points, Burnside lemma, multiply transitive finite groups, Bell numbers, Stirling numbers of the second kind, symmetry group  $S_N$ , Mathieu groups, permutation representations.

MSC-class: 20B20 (primary); 20Cxx, 20C35 (secondary)

# 1 Introduction

In 1975 Goldman [1] proved the following equality, by using interesting statistical (!) arguments:

**Theorem 1.** *The following equality holds for the symmetric group  $S_N$ :*

$$\langle f_{\mathbb{Z}_N}^k \rangle := \frac{1}{|S_N|} \sum_{g \in S_N} f_{\mathbb{Z}_N}(g)^k = \sum_{j=1}^{\min(N,k)} S(k, j), \quad \text{for } k, N \geq 1, \quad (1)$$

where  $S(k, j)$  are the Stirling numbers of the second type.

**Definition.** The **Stirling numbers of the second kind**  $S(k, j)$  count the number of ways to partition the set  $\mathbb{Z}_k = \{1, 2, \dots, k\}$  into  $j$  nonempty subsets. For example, the three integers in  $\mathbb{Z}_3 = \{1, 2, 3\}$  can be separated into  $j = 2$  subsets in  $3 = S(3, 2)$  different ways:

$$\{\{1\}, \{23\}\}, \{\{2\}, \{13\}\}, \{\{3\}, \{12\}\}.$$

The  $S(k, j)$  can be calculated by using their **generating function** [2]:

$$x^k = \sum_{j=0}^k S(k, j)(x)_j = S(k, 0) + \sum_{j=1}^k S(k, j) x(x-1) \cdots (x-j+1). \quad (2)$$

where  $k \in \mathbb{N}$  and  $(x)_j$  is the **falling factorial**, with  $(x)_0 = 1$ . It follows from (2) that  $S(k, 0) = \delta_{k,0}$ . Hence, by substituting  $x = N \in \mathbb{N}$  in (2) we obtain for  $k \geq 1$ :

$$N^k = \sum_{j=1}^k S(k, j)(N)_j, \quad \forall \quad N, k \geq 1. \quad (3)$$

where

$$(N)_j := N(N-1) \cdots (N+1-j) = \frac{N!}{(N-j)!}, \quad \text{for } 1 \leq j \leq N. \quad (4)$$

Note that  $(N)_j := 0$  for  $j > N$ , and  $(N)_{(N-1)} = (N)_N = N!$ .

For  $k \leq N$  Eq. (1) reduces to [3, 4]:

$$\langle f_{\mathbb{Z}_N}^k \rangle := \frac{1}{|S_N|} \sum_{g \in S_N} f_{\mathbb{Z}_N}(g)^k = B_k, \quad \text{for } N \geq k \geq 1, \quad (5)$$

where  $B_k$  denotes the Bell numbers which are related to  $S(k, j)$  by [2]

$$B_k := \sum_{j=1}^k S(k, j), \quad \text{for } k \geq 1. \quad (6)$$

A second generalization of Eq. (5) was obtained by extending the validity of (5) to general  $t$ -transitive groups  $G$  instead of just the symmetric group  $S_N$ :

**Theorem 2.** [5, 6]  *$G$  is  $t$ -transitive on  $X$ , if and only if*

$$\frac{1}{|G|} \sum_{g \in G} f_{\mathbb{Z}_N}(g)^k = B_k, \quad \text{for } k \leq t \leq N := |X|. \quad (7)$$

The equality (7) was first proved by Merris and Pierce (1971) [5] by induction on  $k$ . A second proof was given by Monro and Taylor (1978) [6], by mapping subsets of  $\mathbb{Z}_N^k$  onto partitions of  $\mathbb{Z}_k$ .

In present paper I give in theorem 5 a new proof of (1), based on Burnside lemma [7]. My proof also shows that the r.h.s. of (1) is equal to the number of orbits of the action  $(S_N, \mathbb{Z}_N^k)$ ; this interpretation cannot be deduced from the proof of Goldman [1]. This interpretation is important, since it enables me to give a simple proof of Eq. (15) below, which is valid for any finite group  $G$  and also for  $k > N$ .

## 2 New proofs and results

**Definition.** *When a group  $G$  acts on a  $G$ -set  $Y$ , it decompose it into disjoint orbits. In particular, when  $S_N$  acts on  $Y = \mathbb{Z}_N^k$ , it produces  $S_N$ -**orbits** of different types, as follows:*

$$O_{j,k,N} := S_N \cdot \mathbf{b}_{jk} \in \mathbb{Z}_N^k, \quad j \leq \min(k, N), \quad (8)$$

where  $\mathbf{b}_{jk}$  denotes **basis ordered**  $k$ -tupels which depend on  $j$  distinct integers from  $\mathbb{Z}_j = \{1, 2, \dots, j\}$  (not from  $\mathbb{Z}_N$ )

$$\mathbf{b}_{jk} := (x_1, \dots, x_k) \in \mathbb{Z}_j^k, \quad \text{where } x_i \in \mathbb{Z}_j.$$

**Example.** To illustrate the above notation, consider an  $S_N$ -orbit:

$$\begin{aligned} O_{3,4,N} &:= S_N \cdot \mathbf{b}_{34} = S_N \cdot (1, 2, 3, 2) = \{(g(1), g(2), g(3), g(2)) | g \in S_N\} \\ &= \{(i, j, k, j) | i \neq j \neq k \neq i \in \mathbb{Z}_N\}. \end{aligned}$$

**Theorem 3.** The number of orbits created by the group action  $(S_N, \mathbb{Z}_N^k)$  is given by

$$\mathcal{N}_{orbits}(S_N, \mathbb{Z}_N^k) = \sum_{j=1}^{\min(k,N)} S(k, j), \quad \text{for } k, N \geq 1, \quad (9)$$

*Proof.* The length of an orbit  $O_{j,k,N}$  is independent on  $k$ :

$$|O_{j,k,N}| = |O_{j,j,N}| = (N)_j := N(N-1) \cdots (N+1-j), \quad j \leq N, \quad (10)$$

Let  $n(k, j)$  denotes the number of orbits of type  $O_{j,k,N}$ . It follows that

$$N^k = \sum_{j=1}^{\min(k,N)} n(k, j) |O_{j,k,N}| = \sum_{j=1}^{\min(k,N)} n(k, j) (N)_j. \quad (11)$$

By comparing Eq. (11) with the equality Eq. (3), starting from  $N = 1$  and successively  $N = 2, 3, \dots$ , we obtain

$$n(k, j) = S(k, j), \quad \forall \quad k \geq j \geq 1. \quad (12)$$

□

**Lemma 4.** Let  $f_X(g)$  and  $f_{X^k}(g)$  denote the number of fixed points of the actions  $(G, X)$  and  $(G, X^k)$ , respectively. Then

$$\langle f_X^k \rangle = \langle f_{X^k} \rangle = \mathcal{N}_{orbits}(G, X^k). \quad (13)$$

*Proof.* We recall that  $f_{X^k}(g)$  is equal to the number of ordered  $k$ -tuples  $(x_1, \dots, x_k) \in X^k$  which are fixed by the action of  $g \in G$ . Hence, the first equality in (13) follows from

$$\begin{aligned} f_{X^k}(g) &= \sum_{(x_1, \dots, x_k) \in X^k} \delta_{(x_1, \dots, x_k), g \cdot (x_1, \dots, x_k)} \\ &= \prod_{j=1}^k \left( \sum_{x_j \in X} \delta_{x_j, g \cdot x_j} \right) = f_X(g)^k, \end{aligned} \quad (14)$$

since the summations over  $x_j$  can be carried out independently. The second equality follows immediately from Burnside's lemma [7] for  $(G, X^k)$ . □

Below I generalize the two equalities Eq. (1) and Eq. (7), as follows:

**Theorem 5.** *The group action  $(G, \mathbb{Z}_N)$  is  $t$ -transitive, if and only if the following equality holds:*

$$\frac{1}{|G|} \sum_{g \in G} f_{\mathbb{Z}_N}(g)^k = \mathcal{N}_{orbits}(G, \mathbb{Z}_N^k) \quad (15a)$$

$$= \sum_{j=1}^{\min(k, N)} d_j(G) S(k, j), \quad (15b)$$

$$\Rightarrow \begin{cases} B_k, & \text{for } k \leq t \leq N, \\ B_t + \sum_{j=t+1}^{\min(k, N)} d_j(G) S(k, j) & \text{for } t < k, \end{cases}$$

where the divisions  $d_j(G)$  depend on the group  $G$  and on  $j$ , but not on  $k$ :

$$\begin{aligned} d_j(G) &= 1 \quad \text{for } 1 \leq j \leq t, \quad \text{and} \\ d_j(G) &\geq 2 \quad \text{for } j > t. \end{aligned} \quad (16)$$

*Proof.* Eq. (15a) follows from Eq. (13).

Eq. (15b) follows from Eq. (9) after taking into account:

- Every  $G$  which acts on  $\mathbb{Z}_N$  must be a subgroup of  $S_N$ .
- If  $G$  is a genuine subgroup of  $S_N$ , it would have less group elements. Therefore, we expect

$$|G \cdot \mathbf{b}_{jk}| \leq |S_N \cdot \mathbf{b}_{jk}|. \quad (17)$$

- The equality sign in (17) ( $d_j(G) = 1$ ) holds for  $j \leq t$ , since a  $t$ -transitive group produces exactly the same orbits  $O_{j,k,N}$ , as  $S_N$ , for  $j \leq t$ .
- For  $j > t$  then  $|G \cdot \mathbf{b}_{jk}| < |S_N \cdot \mathbf{b}_{jk}|$ , which means that the corresponding (maximal) orbit  $O_{j,k,N}$  of  $S_N$  must split into  $d_j(G) \geq 2$  orbits of  $G$ .

□

**Example.** The Mathieu group  $M_{24}$  is 5-transitive. Its divisions  $d_j$  can be read from the following formula. (Note that since  $S(k, j) = 0$  for  $k < j$ , the formula (18) is valid for all  $k \geq 1$ .)

$$\begin{aligned} \langle f_{\mathbb{Z}_{24}}^k \rangle = & \sum_{j=1}^5 S(k, j) + 2 S(k, 6) + 9 S(k, 7) + 123 S(k, 8) \\ & + 1938 \sum_{j=9}^{\min(k, 24)} \frac{15!}{(24-j)!} S(k, j). \end{aligned} \quad (18)$$

We can easily understand why the maximal  $S_{24}$ -orbit  $O_{6,6,24}$  has to split:  $|O_{6,6,24}| = (24)_6 = (24)_5 \cdot 19$  is not a divisor of  $|M_{24}| = (24)_5 \cdot 48$ . Hence, I conclude that  $O_{6,6,24}$  must split into two sub-orbits, with lengths  $(24)_5 \cdot 16$  and  $(24)_5 \cdot 3$ , which are divisors of  $|M_{24}|$ . Note that  $d_7 = 9$  and  $d_8 = 123$  yield more predictions.

### 3 Summary

I gave a new proof of Eq. (1) by using the generating function of  $S(k, j)$  and Burnside lemma. Unlike the statistical proof of Goldman [1], my proof led to the equality (15a), between the r.h.s. of (1) to the number of orbits of  $(S_N, \mathbb{Z}_N)$ . This made it possible to derive Eq. (15), which is a generalization of (7) to  $k > N$ , by using a simple argument which led to the inequality in Eq. (17).

Note that a finite group action  $(G, \mathbb{Z}_N)$  yields an  $N \times N$ -matrix representation of  $G$ , which is called **permutation representation**  $\Gamma^P$ , so that the action  $(G, \mathbb{Z}_N^k)$  yields a  $k$ -fold tensor product of  $\Gamma^P$ .

A detailed version of the present paper is available as a preprint, which includes applications and a review of basic concepts of group action. I will gladly send it by email upon request.

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